niveau lines at intervals of 1 c.m. were designated. A rather more elaborate mechanism is necessary to mark the lines of flow v=constant. These lines are so drawn that their vertical projections on the x, y plane are orthogonal to the niveau curves u=constant.

The model, thus constructed, was sent to Hrn. Kreittmäyr in Munich, for duplication. During the summer of 1899 I worked for several days in his workshops but didn't learn enough to make a complete report of this interesting process.

The elliptic integral of the second kind is defined as the logarithmic derivative of the sigma function. The procedure for this function was essentially the same as for the sigma function, but here four parallelograms of periods are sufficient, as the function is periodic with regard to one of the periods of the elliptic function. The three theorems for sigma still hold for this function.

One fact is at once apparent from the model of the sigma function; for values of z within the first parallelogram it is of practical value in numerical calculations, but comparatively cumbersome for the next parallelogram and utterly worthless for parallelograms still further removed.

Both models are published by Martin Schilling, in Halle, successor to S. Brill, in Darmstadt.

MERIDIAN AND TRANSVERSE SECTIONS OF HELICOIDS OF UNIFORM PITCH.

By ALFRED HUME. C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Miss.

The following discussion was suggested by the treatment of this subject given in MacCord's Descriptive Geometry. In addition to establishing analytically the facts stated in that text-book, other properties of the curves of intersection have been determined.

The helicoid here considered is a surface generated by a straight line always tangent to a right circular cylinder, inclined at a constant angle to the axis of the cylinder, and having two simultaneous uniform motions—one of revolution about the axis, the other of translation parallel thereto. The locus of the point of tangency of this generatrix with the cylindrical surface is a common helix.

Let AB be the axis of the cylinder, C the center of a right section which intersects the helix OH at O, Q any point on OH, RT the corresponding position of the generatrix piercing the plane ABS, determined by the axis and O, at P, and meeting the plane of the right section at T.

Then P is any point of the meridian section of the helicoid and T any point of the transverse section.

A. To Find the Locus of P.

The plane through the generatrix RT and tangent to the cylinder cuts CO produced at E, ET being tangent to the circular section at D.

PE is parallel to QD which is an element of the cylindrical surface.

Draw through Q a parallel to DE, meeting PE at L.

Draw CO and CD.

Denote the angle COD by θ, the radius of the cylinder by a, the pitch of the helix by P, and the angle PTE by α.

Taking CO produced as the X-axis, and OY, parallel to AB, as the Y-axis, $EO=x$, and $PE=y$.

$x=EC-OC=a(\sec\theta-1)$....(1).

$y=PL+QD$.

$PL=LQ\tan\alpha=ED\tan\alpha=a\tan\theta\tan\alpha=c a\tan\theta$, letting $\tan\alpha=c$.

$$QD:P=\theta:2\pi;\quad QD=\frac{\theta}{2\pi}P=K\theta,$$

where $K=\dfrac{P}{2\pi}$.

Therefore, $y=K\theta+ca\tan\theta$....(2).

(1) and (2) are the equations of the meridian section of the helicoid.

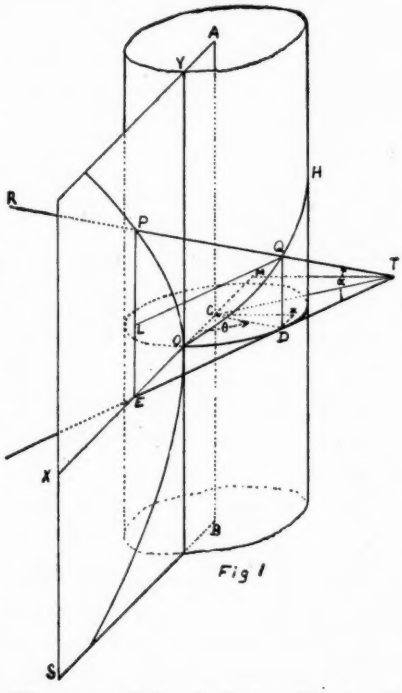


Fig 1

There are certain properties of this curve independent of the value of c. These are deduced first. Afterward, the variations in form and the singularities are noted as c varies from $+\infty$ to $-\infty$.

Changing the sign of θ does not affect x but gives to y values, equal numerically, but of contrary sign. Hence, the curve is symmetrical with reference to the X-axis.

Since when $\theta=0$, $y=x=0$, the curve passes through the origin.

When $\theta=\pm\frac{1}{2}\pi$, $x=\infty$, and y becomes infinite except when $c=0$. There are, therefore, infinite branches.

$$\frac{dy}{dx}=\frac{K+ca\sec^2\theta}{a\sec\theta\tan\theta}=\frac{K\cos^2\theta+ca}{a\sin\theta}\text{....(3).}$$

When $\theta=\frac{1}{2}\pi$, $\dfrac{dy}{dx}=c$....(4), and the Y-intercept of the corresponding tangent$=\dfrac{\pi K}{2}+ca$.

An asymptote is $y=cx+\frac{1}{4}P+ca$....(5), a line crossing AB at a distance

above C equal to one-fourth the pitch and making an angle with the X-axis equal to a.

Equating to zero, $\frac{1}{a}(K\cos^2\theta - 2K - ca)\cot\theta\csc\theta$, the derivative of $\frac{dy}{dx}$ with regard to θ, gives

$$\cos\theta = \pm\sqrt{2+\frac{ca}{K}} \ldots\ldots (6).$$

(a). $c>0$.

From (3), when $\theta=0$, $\frac{dy}{dx}=\infty$; the curve is tangent to the Y-axis at the origin.

From (6), there is no point of inflexion.

The curve is represented in Fig. 2, positive values of θ giving points above the axis on OP; negative, points below on OW. UV is an asymptote, CU being $\frac{1}{4}P$.

(b). $c=0$.

The equation of the asymptote becomes $y=\frac{1}{4}P$, the curve approaching parallellism with the X-axis, and appearing as indicated in Fig. 2 by the accented letters.

(c). $c<0$.

Case I. $c<\frac{K}{a}$, numerically.

It follows from (2) that there is some positive value of θ, less than $\frac{1}{2}\pi$ and other than zero, for which $y=0$. At this point

$$\frac{dy}{dx}=\frac{c}{2}\,\frac{2\theta-\sin 2\theta}{\theta.\sin\theta},$$

which is positive when θ is negative, and negative when θ is positive. Hence the point under consideration is a double point of intersection.

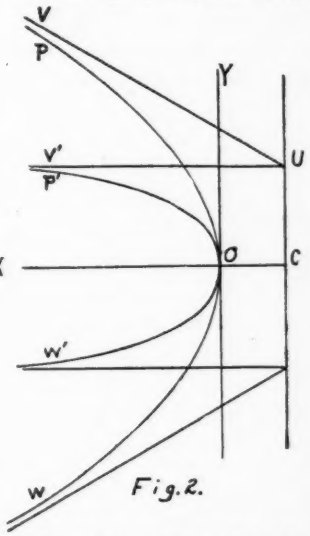


Fig. 2.

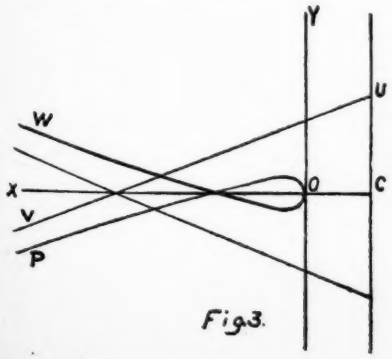


Fig. 3.

Somewhere between it and the origin the curve is parallel to the X-axis, this being the case when $\theta=\cos^{-1}\sqrt{-\frac{ca}{K}}$.

(6) shows that there is no point of inflexion. See Fig. 3.

Case II. $c=\frac{K}{a}$, numerically.

Since $K=-ca$, (2) becomes $y=K(\theta-\tan\theta)$, from which y is negative as θ increases from 0 to $\frac{1}{2}\pi$, and decreases without limit, having neither a maximum nor a minimum value for any value of θ.

When $\theta=0$, (3) assumes the form $\frac{0}{0}$. But it may be written

$$\frac{K(\cos^2\theta-1)}{a\sin\theta}\left(=-\frac{K}{a}\sin\theta\right),$$

which equals zero when $\theta=0$. Hence there is a cusp of the first species at the origin.

(6) gives $\cos\theta=\pm1$, but this does not correspond to a point of inflexion since $\frac{d}{d\theta}\left(\frac{dy}{dx}\right)$ does not change sign as θ passes through 0.

This curve is drawn in Fig. 4.

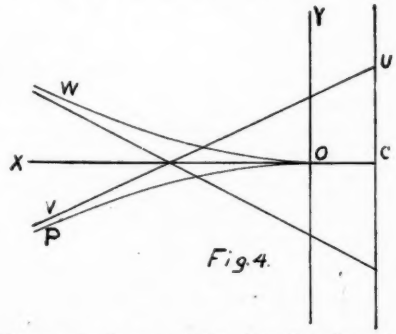


Fig. 4.

Case III. $c>\frac{K}{a}$ but $<\frac{2K}{a}$, numerically.

By (2) y is negative and θ increases from 0 to $\frac{1}{2}\pi$.

By (3) there is no point at which the curve is parallel to the X-axis; at the origin it is tangent to the Y-axis.

By (6) there is a point of inflexion where

$$\theta=\cos^{-1}\sqrt{2+\frac{ca}{K}}.$$

See Fig. 5.

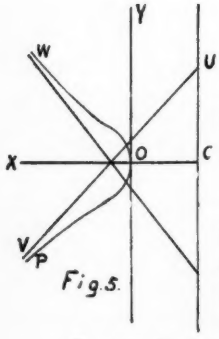


Fig. 5.

Case IV. $c>\frac{2K}{a}$, numerically, but negative.

Points corresponding to θ between 0 and $\frac{1}{2}\pi$ lie below the X-axis. The curve is tangent to the Y-axis at the origin, and, by (6), there is no point of inflexion.

The curve is shown in Fig. 6.

When $c=\infty$, the helicoidal surface becomes the cylindrical, the meridian section coinciding with the Y-axis. As c changes sign, passing through infinity, the portions of the curve, OP and OW, exchange positions with reference to the X-axis.

It may be be noted that in (c) Case II the inclination of the generatrix equals that of the helix, the helicoid becoming the convolute.

Equation (4) shows that the curve at infinity always has the direction of the generatrix when parallel to the meridian plane. This plane remaining fixed, while the inclination of the generatrix varies, the asymptote turns about the stationary point U through 180°, as is evident from equation (5).

B. To Find the Locus of T.

Through T draw a line perpendicular to OC, meeting it, produced, at M. Through C draw a parallel to MT meeting a perpendicular to it from D at Z.

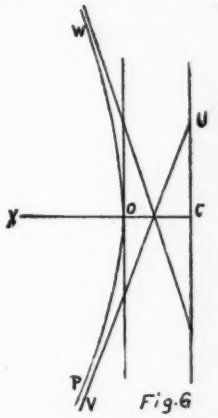


Fig. 6

Taking CO and CZ for the X- and Y-axes, respectively, $MC=x$ and $MT=y$.

Since $DT=\frac{QD}{\tan\alpha}=\frac{K\theta}{c}$,

$$x=a\cos\theta-\frac{K\theta}{c}\sin\theta....(1),$$

$$\text{and } y=a\sin\theta+\frac{K\theta}{c}\cos\theta....(2),$$

from which

$$x^2+y^2=a^2+\frac{K^2\theta^2}{c^2},\ \text{or, } r^2=a^2+\frac{K^2\theta^2}{c^2},\ (r=CT)....(3).$$

These are the equations of the transverse section. They show—(1) and (2) together, or (3) alone—that the curve is symmetrical with respect to the X-axis; it is a spiral beginning at O and cutting the axes of X and Y, in general, an infinite number of times at points more and more remote from C.

As in the case of the meridian section, this locus has distinguishing features depending on the value of C. These are shown in Fig. 7, a part of the first spiral only being represented. The loci are traced for values of θ between 0 and π; also, symmetrical curves, OX being the axis of symmetry, for numerically equal negative values of θ.

$$\frac{dy}{dx}=\frac{K\theta\tan\theta-(ca+K)}{K\theta+(ca+K)\tan\theta}=\tan(\theta-\phi)....(4)$$

where $\phi=\tan^{-1}\frac{ca+K}{K\theta}$.

$$\frac{d}{d\theta}\left(\frac{dy}{dx}\right)=\sec^2(\theta-\phi)\,\frac{K^2\theta^2+c^2a^2+3caK+2K^2}{K^2\theta^2+(ca+K)^2}....(5).$$

(a). $c>0$.

The curve is perpendicular to the X-axis at O, is concave toward C, and has no point of inflexion. In Fig. 7, OTI is a part of a spiral of this class.

(b). $c=0$.

The transverse section coincides with the generatrix when $\theta=0$. In Fig. 7, it is shown as OY', parallel to the Y-axis.

(c). $c<0$.

Case I. $c<\frac{K}{a}$, numerically.

While the X-axis is still crossed at right angles at O, y is negative until θ reaches a value given by the equation

$$\tan\theta+\frac{K}{ca}\theta=0.$$

By (4), the slope is first negative and, by (5) it increases with θ.

Hence, there is a double point of intersection as seen in $OT2$.

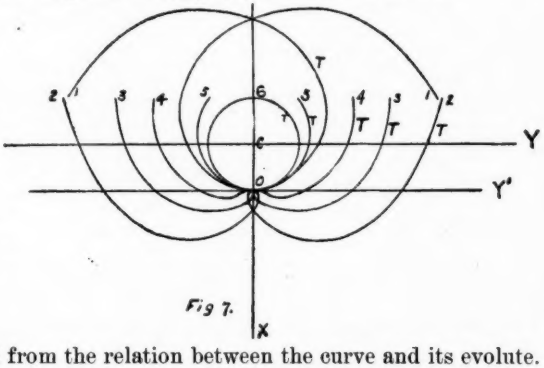


Fig 7.

Case II. $c=\frac{K}{a}$, numerically.

(3) reduces to $r^2=a^2+a^2\theta^2$, the transverse section becoming the involute of the circular section.

(4) gives $\frac{dy}{dx}=\tan\theta$, as might have been inferred from the relation between the curve and its evolute.

When $\theta=0$, $\frac{dy}{dx}=0$, and there is a cusp of the first species at O. There being no point of inflexion, the locus appears as in $OT3$.

Case III. $c>\frac{K}{a}$ but $<\frac{2K}{a}$, numerically.

Again the X-axis is intersected perpendicularly at O.

Equating (5) to zero,

$$\theta=\sqrt{-\left(\frac{ca}{K}+1\right)\left(\frac{ca}{K}+2\right)}.$$

Hence there is a point of inflexion. See $OT4$.

Case IV. $c>\frac{2K}{a}$, numerically, but negative.

There is no point of inflexion. The curve is perpendicular to the X-axis at O and is concave toward C. $OT5$ represents it.

When $c=\infty$, (1) and (2) become $x=a\cos\theta$ and $y=a\sin\theta$, respectively; while (3) reduces to $K^2+y^2=a^2$, or, $r=a$. The transverse section coincides with the right circular section. See $OT6$.

The curve intersects the line OY' for some value of θ less than π when c lies between 0 and $-\frac{2K}{a}$. That this is true may be seen by making $x=a$ in (1), obtaining, after reduction,

$$\tan\tfrac{1}{2}\theta = -\frac{K'}{ca}\,\theta.$$

If $a=0$, (3) becomes $r=\pm\frac{K}{c}\theta$, the equation of a spiral of Archimedes.

Under this hypothesis, O and D (Fig. 1) coincide with C.

CT, coinciding with DT, is perpendicular to CD, and, since OCZ is a right angle, $\angle TCZ = \angle OCD = \theta$.

Hence CZ is the initial line and C the pole.

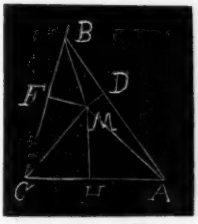
PROVING THE FALSE.

By DR. GEORGE BRUCE HALSTED.

I. THEOREM. Every triangle is isosceles.

HYPOTHESIS. Let ABC be a triangle.

CONCLUSION. $AC=BC$.



PROOF. Take the bisector of $\measuredangle C$, and erect the perpendicular bisector of AB.

From their intersection M drop the perpendiculars MF, MH. Then $\triangle AMD \equiv \triangle BMD$ (by Euclid I. 4).

$\therefore AM=BM$.

But $\triangle CMH \equiv \triangle CMF$ (by Euclid I. 26). $\therefore MF=MH$, and $CF=CH$.

$\therefore \triangle AMH \equiv \triangle BMF$ (by Halsted's *Elements*, 179).

$\therefore BF=AH$. But already we have $FC=HC$.

Therefore $BC=AC$. Q. E. D.

II. Just as false is the following from Olney's *Geometry*, Section VIII, Proposition XIV.

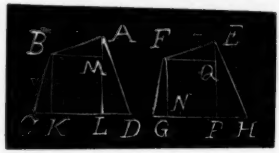
THEOREM. "Two quadrilaterals having three sides of the one equal to the the three corresponding sides of the other, each to each, and the two corresponding angles adjacent to the unknown sides equal, each to each, are equal figures."

A simple construction, due to my pupil R. L. Moore, shows that they may be as *unequal* as we please.

Make DG greater than DC by whatever sect you please, say a centimeter. Take $\measuredangle DGF = \measuredangle DCB$. Take $GF=CB$. Take A on the perpendicular bisector of BF. Then the quadrilaterals $ABCD$, $AFGD$ fulfill the hypothesis of the theroem.

My friend G. B. M. Zerr, in a moment of distraction, has given the following as proof that these unequals are equal, and it seems that Mr. J. Scheffer agrees therein. See THE AMERICAN MATHEMATICAL MONTHLY, April, 1902, Vol. IX, page 105.

Let $ABCD$, $EFGH$ be the two quadrilaterals; $BC=FG$, $AB=EF$, $AD=EH$, $\measuredangle BCD=\measuredangle FGH$, $\measuredangle ADC=\measuredangle EHG$. Draw BK, AL perpendicular to CD; FN, EP perpendicular to GH; BM perpendicular to AL; FQ perpendicular to EP.



Right triangles BCK and FGN are equal, also right triangles ALD and EPH (by Euclid I. 26).

$\therefore$ $BK=FN$, $AL=EP$, also $AL-BK=AM=EP-FN=EQ$.

$\therefore$ right triangles $ABM=FEQ$; since $AB=FE$ and $AM=EQ$.

$\therefore$ $BM=FQ$. $\therefore$ $BM=KL=FQ=NP$. $\therefore$ $BKLM=FNPQ$.

$\therefore$ $BCK+ADL+ABM+BKLM=FGN+EPH+FEQ+FNPQ$.

$\therefore$ $ABCD=EFGH$, the part equal to the whole. Q. E. D.

Now what is the explanation of these proofs of the false? Not exactly that for the extraordinary blunder which used to stand on page 224 of Wentworth's *Geometry* (for ten years at least from 1877 to 1887), Proposition XIII, §387. "To inscribe a regular polygon of any number of sides in a given circle."

This, like the blunder of McLellan and Dewey, in making number the outcome of measurement, could only coexist with a colossal ignorance of the history of mathematics, while pseudo-proofs like Professor Zerr's, which may be characterized as figure proofs, are given by those who, like Prof. D. E. Smith, need not be supposed unusually ignorant of the history of mathematics.

A figure we should use, but never suppose that we may rest our proof upon it, draw our proof from supposed perception of it. We must always take care that the operations supposed carried out on a figure retain a pure logical validity and inter-connection.

The figure is a help, which, this forgotten, may become a harm.

Moreover, what are called *problems of construction* have a double import. Theoretically, they are really theorems declaring that the existence of certain points, sects, straights, angles, circles, etc., follows logically by rigorous deduction from the existences postulated in our assumptions. Thus the possibility of solving such problems by elementary geometry is a matter absolutely essential in the logical sequence of our theorems.

So, for example, elementary geometry shows (Halsted's *Elements*, §502) that a sect has always trisection points, and this may be expressed by saying we have solved the problem to trisect any sect. Now it happens that a solution of the problem to trisect any angle is impossible with only the assumptions of elementary geometry. Thus any reference to or use of results following from or dependent upon the trisection of the angle would be equivalent to the introduction of additional assumptions or postulates.

Thus the whole "Note on Assumed Constructions" in Beman and Smith's *Geometry*, 1899, page 70, §112, is seen to be as erroneous as Wentworth's problem and Professor Zerr's proof.

Austin, Texas, May 3, 1902.

NOTE. Professor Norris, the proposer of Problem 169, Geometry, over a year ago sent with the supposed theorem a note calling attention to the fact that it was not true in

general. This note got lost, and when the selection of proofs was made, the fact of the theorem being false had slipped my mind.

We are glad that Dr. Halsted has contributed this article, as it will help to keep geometricians on the straight road to rigorous demonstrations. His article on "Some Fallacies in Wentworth's Geometry" has awakened quite an interest among teachers in the high schools of the country.

By his fearless exposition of fallacies in the teaching of Elementary Geometry, Dr. Halstead has done more to improve geometrical instruction in America, perhaps, than any other American mathematician.

However, we do not entirely agree with him in his criticism on the "Note on Assumed Constructions" in Beman and Smith's Geometry, Second Edition. For the benefit of those of our readers who may not have a copy of the book at hand, we quote the note in full as follows:

"It has been assumed, up to Proposition XXVIII, that all constructions were made as required for the theorems. Thus an equilateral triangle has been frequently mentioned, although the method of constructing one has not been indicated; a regular heptagon has been mentioned in Ex. 93, and reference might be made to certain results following from the trisection of an angle, although the solutions of the problems, to construct a regular heptagon and to trisect an angle, are impossible by elementary geometry. But the possibility of solving such problems has nothing to do with the logical sequence of the theorems. One may know that each angle of a regular heptagon is $5\text{-}7\times180°$, whether the regular heptagon admits of construction or not. Nevertheless, an important part of geometry concerns itself with the construction of certain figures—a part of most practical value and of much interest to the student of mathematics."

We can see nothing erroneous in this note. If it is proved by perfectly rigorous and elementary methods that the sum of the interior angles of any convex polygon of n sides is $(n-2)$ straight angles, it certainly follows that each angle of a regular heptagon is 5-7 of a straight angle, whether the heptagon can be constructed by elementary methods or not; or whether, indeed, a regular heptagon can be constructed by any method whatever. So, too, if we are given an angle of $33°$, $\frac{1}{3}$ of that angle is $11°$, whether the angle can be trisected or not. In this we agree with Professors Beman and Smith. But on the other hand we agree with Dr. Halsted in protesting against requiring constructions of such Propositions as the one referred to in Wentworth's Geometry, viz: "To inscribe a regular polygon of any number of sides in a given circle." Such propositions are misleading alike to teacher and student. We do not remember of having come to us, a student having completed his plane geometry elsewhere who had the slightest suspicion that it is any more difficult to inscribe a regular heptagon in a circle than it is to inscribe a pentagon. In looking through some of the geometries in our possession, we find a few others beside Wentworth who have to plead guilty to this fault. Wentworth has left the proposition referred to out of the last edition of his geometry.

Now as to problem 169, taken from Olney's Geometry, Section VIII. We find on consulting Olney's Geometry, University Edition, 1881, that the theorem as given by Prof. Olney in this edition was not correctly quoted. The theorem, in part, reads as follows: *Two quadrilaterals are equal when the following parts are equal, each to each, in both quadrilaterals, and similarly arranged:* 1 ... 2.... 3.... 4. *The three sides and two included angles.* This theorem is definite.

In Olney's Elementary Geometry, edition of 1883, the following proposition is given on page 144: Proposition XII. A quadrilateral is determined when there are given in their order: 1.... 2.... 3. 1st, Three sides and two included angles. 2d, When the two angles are not both included between the known sides, the case may be ambiguous.

In discussing this second case, Olney draws a figure exactly as Dr. Halsted has, thus showing that the proposition is not generally true. We are not aware that this proposition is given in *any* geometry, as published in the MONTHLY.—EDITOR.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

156. Proposed by JAMES F. LAWRENCE, A. B., Professor of Mathematics, Rogers Academy, Rogers, Ark.

Suppose that in a meadow the grass is of uniform quality and growth, and that 6 oxen or 10 colts could eat up 3 acres of pasture in 18-25 of the time in which 10 oxen and 6 colts could eat up 8 acres; or that 600 sheep would require 2 6-7 weeks longer than 660 sheep to eat 9 acres. In what time could 1 ox, 1 colt and 1 sheep eat up 1 acre of pasture, on the supposition that 588 sheep eat as much in a week as 6 oxen and 11 colts?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let the standing grass on one acre be denoted by 1, and the weekly growth on one acre by u. From the first condition: Because 6 oxen eat all that grows on 3 acres each week and in addition $25/18t$ part of the grass standing on 3 acres; and because 1 colt eats $\frac{3}{5}$ times as much as 1 ox, we have what 1 ox eats in 1 week $=\frac{1}{6}(3\times\frac{25}{18t}+3u)=\frac{1}{36t}(25+18tu)$....(1), what 1 colt eats in 1 week $=\frac{1}{60t}(25+18tu)$....(2).

From the second condition we have: What 10 oxen and 6 colts eat in 1 week $=8/t+8u$....(3). We have considered $\frac{1}{2}\frac{8}{5}t$ weeks in first condition, and t weeks in second condition as the time required.

From (1), (2), (3), we have $\frac{1}{90t}(25+9)(25+18tu)=\frac{8}{t}(1+tu)$, or $(25+9)(25+18tu)=720(1+tu)=34(25+18tu)$....(4).

Suppose that 9 acres will support 660 sheep for x weeks; then 1 sheep in 1 week will eat $\frac{3}{220x}(1+ux)$....(5). Also from first statement, 1 sheep will eat $\frac{1}{8\,6\,0}(\frac{9}{x+2\frac{6}{7}}+9u)=\frac{3}{2\,6\,0}(\frac{7}{7x+20}+u)$.... (6).

Now 3 oxen=5 colts=140 sheep.

From (1) and (5), $11x(25+18ut)=252t(1+ux)$....(7).

From (5) and (6), $7ux^2+20ux+7x=200$....(8).

From (4), $65=54tu$, or $u=65154t$. (7) becomes, by substituting u, $5x=6t$, or $t=5x/6$.

$\therefore u=13/9x$. This in (8) gives $x=10$ weeks.

$\therefore u=\frac{1\,3}{9\,0}$, $t=\frac{2\,5}{3}$.

From (1), 1 ox eats in 1 week, $\frac{7}{4\,5}$;

From (2), 1 colt eats in 1 week, $\frac{7}{7\,5}$;

From (5), 1 sheep eats in 1 week, $\frac{1}{3\,6\,0}$.

$\frac{7}{4\,5}+\frac{7}{7\,5}+\frac{1}{3\,6\,0}=\frac{2\,8}{9\,0\,0}\frac{7}{0}$, what 1 ox, 1 colt and 1 sheep eat in 1 week.

The grass on 1 acre in 1 week with what grows $=1+\frac{13}{90}=\frac{103}{90}$.
$\frac{273}{900}-\frac{13}{90}=\frac{97}{900}$, the amount of the standing grass eaten in 1 week.
$1\div\frac{97}{900}=\frac{900}{97}=9\frac{27}{97}$ weeks, the time required.

SOME INTERESTING RULES IN MULTIPLICATION.

BY MARY M. CURRIER, WENTWORTH, N. H.

RULE 1. To multiply one number by another, the multiplier consisting of two digits of which the left-hand digit is 1:

Multiply each figure of the multiplicand by the right-hand figure of the multiplier, and to each product add the figure of the multiplicand following the one multiplied.

Example 1. Multiply 1675 by 13.

$1675\times13=21775$

$5\times3=15$
$7\times3=21,\ 21+1+5=27$
$6\times3=18,\ 18+2+7=27$
$1\times3=3,\ 3+2+6=11$
$1+1=2$

Taking the digits in the units place in these several products beginning with the last, we have, $21775=13\times1675$.

Example 2. Multiply 40928 by 17.

$8\times7=56$
$2\times7=14,\ 14+5+8=27$
$9\times7=63,\ 63+2+2=67$
$0\times7=0,\ 0+6+9=15$
$4\times7=28,\ 28+1+0=29$
$2+4=6$

$\therefore\ 695776=17\times40926$.

RULE 2. To multiply one number by another, the multiplier consisting of three digits, of which the two at the left are ones:

Multiply each figure of the multiplicand by the right hand figure of the multiplier and to each product add the two figures following the one multiplied and the digit in the tens place of the preceding product.

Example 1. Multiply 340726 by 114.

$6\times4=24$
$2\times4=8,\ 8+2+6=16$
$7\times4=28,\ 28+1+2+6=37$
$0\times4=0,\ 0+3+7+2=12$
$4\times4=16,\ 16+1+0+7=24$
$3\times4=12,\ 12+2+4+0=18$
$1+3+4=8$
$3=3$

$\therefore\ 38842764=114\times340726$.

RULE 3. To multiply one number by another, the multiplier consisting of two digits of which the left hand digit is 2.

Multiply each figure of the multiplicand by the right-hand figure of the multiplier and to each product add twice the figures of the multiplicand following the one multiplied and the digit in tens place of the preciding product.

Example 1. Multiply 3495013 by 26.

$$\begin{aligned}
&3\times6 && =18\\
&1\times6=\ 6, && 6+1+2\times3=13\\
&0\times6=\ 0, && 0+1+2\times1=\ 3\\
&5\times6=30, && 30+0+2\times0=30\\
&9\times6=54, && 54+3+2\times5=67\\
&4\times6=24, && 24+6+2\times9=48\\
&3\times6=18, && 18+4+2\times4=30\\
& && 3+2\times3=\ 9
\end{aligned}$$

$\therefore$ $90870338=26\times3495013$.

From these three rules one may see how the principle may be carried further.

NOTE. We deem these rules of sufficient interest to merit publication in this department. It may be seen that they are results of a compact arrangement of the following method: Multiply 3492 by 117.

3492	2×7 $=14$
117	$9\times7=63,\ 63+1+2$ $=66$
—	$4\times7=28,\ 28+6+9+2=45$
14	$3\times7=21,\ 21+4+4+9=38$
63.	$3+3+4=10$
28..	$1+\ 3=4$
21...	
3492.	$\therefore$ $408564=117\times3492$.
3492..	
—	EDITOR.
408564	

ALGEBRA.

139. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Solve neatly $\sqrt[4]{(m-x)}=\sqrt[4]{n}-\sqrt[4]{x}$.

Solution by J. H. DRUMMOND, LL. D., Portland, Me.

$(m-x)^{\frac{1}{4}}=n^{\frac{1}{4}}-x^{\frac{1}{4}}$ ....(1).

Put $n=p^4$ and $x=y^4$. Substituting these values in (1), raising to the fourth power, and reducing, we have

$$y^4-2py^3+3p^2y^2-2py^3=\frac{m-p^4}{2}.$$

By inspection we see that adding p^4 to the first member it becomes a perfect square. Adding p^4 to both members and extracting the square root we have

$$y^2-py+p^2=\pm\sqrt{\frac{m+p^4}{2}};\ \text{then}\ y=\frac{p\pm[-3p^2\pm2\sqrt{(2m+2p^4)}]^{\frac{1}{2}}}{2}.$$

Restoring values of y and p, we have

$$x=\left(\frac{n^{\frac{1}{4}}\pm[-3n^{\frac{1}{2}}\pm 2\sqrt{(2m+2n)}]^{\frac{1}{2}}}{16}\right)^4.$$

It is a pretty good question in Diophantine Analysis to give such values to m and n as will make x a rational whole number. If $m=17$, and $n=81$, $x=16$ or 1. But if we go back to the original equation, it becomes very easy for n and x may be any fourth powers, say p^4 and q^4, and $m=(p-q)^4+q^4$.

Solved similarly by *G. B. M. ZERR*. F. P. Matz solved it by making $x=m\sin\phi$.

140. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

A man pays monthly \$24.50 for 8 years for a loan of \$1250. What is the rate %?

Solution by the PROPOSER.

Let $12r$=rate %.

$$\therefore\ 24.50=\frac{1250r(1+r)^{96}}{(1+r)^{96}-1}.$$

$$\therefore\ (2500r-49)(1+r)^{96}=49.\quad \therefore\ r=.02203 \text{ nearly.}\quad 12r=26.43\%.$$

141. Proposed by JOSEPH V. COLLINS, Ph. D., Professor of Mathematics, State Normal School, Stevens Point, Wis.

How many teams of two horses each can a livery stable man send out who has 10 horses, assuming (1) that we consider the way the team is hitched and (2) that we do not.

Suppose he has 8 horses; 10 horses. Suppose he has 7 buggies, then how many different rigs can he send out, assuming that he has 10 horses, and counting both one and two horse rigs?

No solution of this problem has yet been received.

142. Proposed by A. H. BELL, Hillsboro, Ill.

If x/y is the convergent preceding the complete quotient $(\sqrt{A}+m)/n$; prove that $x^2-Ay^2=\pm n$.

Solution by H. S. VANDIVER, Bala, Pa.

Expand $\sqrt{A}$ in a continued fraction. Let P_k/Q_k denote the convergent preceding $\dfrac{\sqrt{A}+m}{n}$, and let $\dfrac{P_{k-1}}{Q_{k-1}}$ denote the convergent immediately preceding P_k/Q_k, then

$$\sqrt{A}=\frac{P_kx_k+P_{k-1}}{Q_kx_k+Q_{n-1}} \text{ where } x_k=\frac{\sqrt{A}+m}{n}.$$

Substituting this value of x_k, ahd simplifying,

$$\sqrt{A}=\frac{P_n(\sqrt{A}+m)+kP_{n-1}}{Q_n(\sqrt{A}+m)+nQ_{n-1}}.$$

Multiplying out, and equating rational and irrational parts, there is obtained

$$Q_k m+nQ_{k-1}=P_k....(1),$$
$$P_k m+nP_{k-1}=Q_k A....(2).$$

Solving for n in (2) and making use of the relation $P_kQ_{k-1}-Q_kP_{k-1}=(-1)^k$ we get, $P_k^2-AQ_k^2=(-1)^k n$.

Also solved by *G. B. M. ZERR*.

143. Proposed by **JOHN M. COLAW, A. M., Monterey, Va.**

Solve $x+y+z+u=a....(1)$.
$x^2+y^2+z^2+u^2=b....(2)$.
$x^3+y^3+z^3+u^3=c....(3)$.
$x^4+y^4+z^4+u^4=d....(4)$.

Solution by **MARCUS BAKER, Washington, D. C.**

Let u, x, y and z be the roots of the equation

$$X^4-AX^3+BX^2-CX+D=(X-u)(X-x)(X-y)(X-z).$$

For X write $1/V$, whence

$$1-AV+BV^2-CV^3+DV^4=(1-uV)(1-xV)(1-yV)(1-zV);$$

whence $\log(1-AV+BV^2-CV^3+DV^4)$

$$\begin{aligned}&=\log(1-uV)+\log(1+xV)+\log(1-yV)+\log(1-zV).\\&=-uV-\tfrac{1}{2}u^2V^2-\tfrac{1}{3}u^3V^3-\tfrac{1}{4}u^4V^4-....\\&\quad -xV-\tfrac{1}{2}x^2V^2-\tfrac{1}{3}x^3V^3-\tfrac{1}{4}x^4V^4-....\\&\quad -yV-\tfrac{1}{2}y^2V^2-\tfrac{1}{3}y^3V^3-\tfrac{1}{4}y^4V^4-....\\&\quad -zV-\tfrac{1}{2}z^2V^2-\tfrac{1}{3}z^3V^3-\tfrac{1}{4}z^4V^4-....\\&=-aV-\tfrac{1}{2}bV^2-\tfrac{1}{3}cV^3-\tfrac{1}{4}dV^4-....\end{aligned}$$

whence $1-AV+BV^2-CV^3+DV^4=e^{-(aV+\frac{1}{2}bV^2+\frac{1}{3}cV^3+\frac{1}{4}dV^4+....)}$

Developing the second member of this equation, remembering that

$$e^{-x}=1-x-\frac{x^2}{2!}-\frac{x^3}{3!}-\frac{x^4}{4!}....$$

we have

$$1-AV+BV^2-CV^3+DV^4=1-a\left|\begin{array}{l}V\ -\frac{1}{2}b\\ \quad +\frac{1}{2}a^2\end{array}\right|\left|\begin{array}{l}V^3\ -\frac{1}{3}c\\ \quad +\frac{1}{2}ab\\ \quad -\frac{1}{6}a^3\end{array}\right|\left|\begin{array}{l}V\ -\frac{1}{4}d\\ \quad +\frac{1}{3}ac\\ \quad +\frac{1}{8}b^2\\ \quad -\frac{1}{4}a^2b\\ \quad +\frac{1}{24}a^4\end{array}\right|\left|\begin{array}{l}V^4-....\end{array}\right.$$

Equating like coefficients, we have

$$A=a;\ B=\tfrac{1}{2}(a^2-b);\ C+\tfrac{1}{6}(a^3-3ab+2c);\ D=\tfrac{1}{24}[a^4-6a^2b+(3b^2+8ac)-6d].$$

Therefore u, x, y and z are the roots of the equations

$$X^4-aX^3+\tfrac{1}{2}(a^2-b)X^2-\tfrac{1}{6}(a^3-3ab+2c)X+\tfrac{1}{24}(a^4-6a^2b[8ac+3b^2]-6d)=0.$$

The *method* here used I obtained many years ago from my old friend James Main of the Coast and Geodetic Survey who died in Washington November 23, 1894, aged 84 years. This method is *general* applying to a set of n equations containing n unkown quantities.

If we have

$$x+y+z+....=a$$
$$x^2+y^2+z^2+....=b$$
$$x^3+y^3+z^3+....=c$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$
$$x^n+y^n+z^n+....=k$$

then are x, y, z, etc., the roots of

$$X^n-AX^{n-1}+BX^{n-2}-CX^{n-3}+....\pm K=0,$$

where

$A=a$

$B=(1/2!)\ [a^2-b]$

$C=(1/3!)\ [a^3-3ab+2c]$

$D=(1/4!)\ [a^4-6a^2b+(8ac+3b^2)-6d]$

$E=(1/5!)\ [a^5-10a^3b+5a(4ac+3b^2)-10(3ad+2bc)+24c]$

$F=(1/6!)\ [a^6-15a^4b+5a^2(8ac+9b^2)-15(6a^2d+8abc+b^3)+2(72ae+45bd+20c^2)-120f]$

$G=(1/7!)\ [a^7-21a^5b+35a^3(2ac+3b^2)-105a(2a^2d+3bc+b^3+b^2c)+14(24a^2e+20ac^2+15b^2c+45abd)-84(10af+6be+5cd)+720g]$

Solved in an excellent manner by *G. B. M. ZERR.*

144. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy, Defiance College, Defiance, Ohio.

Show that the number of ways in which 15 different problems may be distributed among 5 students so that each student shall have three of them, is $N=(5.3)!/(3!)^5$.

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa., and M. E. GRABER, Heidelberg University, Tiffin, Ohio.

It is demonstrated that the number of ways $a+b+c+....$ can be divided into groups containing a things in one group, b in another, etc., is

$$\frac{[a+b+c+....]!}{[a!][b!][c!]....}.$$

Let $a=b=c=\text{etc.}=3$.

$\therefore\ n=\dfrac{[5.3]!}{[3!]^5}=\dfrac{15!}{[3!]^5}$.

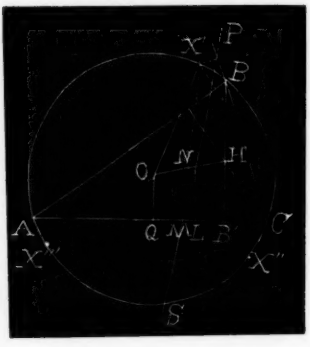
GEOMETRY.

172. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud. Gloucestershire, England.

The center N of the 9-point circle of a triangle ABC lies on P, the pedal line of a point on the circumcircle. Find the angle of intersection of P and AB.

Solution by MARCUS BAKER, Washington, D. C.

In the annexed figure PS is *Simpson's line*, or the pedal line of the circumcircle. Lines drawn from any point X' in the circumcircle perpendicular to the sides of ABC determine three collinear points on Simpson's line. In the figure a point X has been so chosen that the resulting *Simpson line* passes through N, the center of the Twelve Point or Feuerbach circle. H is the orthocenter, O the circumcenter, and N the Feuerbach center. $NO=NH=\frac{1}{2}OH$.



Let φ=the angle NML to be determined.

$$\tan\varphi=\frac{LN}{LM}=\frac{\frac{1}{2}[OQ+HB']}{\frac{1}{2}QB'-QM}$$

$$=\frac{\frac{1}{2}[R\cos B+2R\cos C\cos A]}{\frac{1}{2}[2R\sin C\cos A-R\sin B]-R\cos\varphi}$$

$$\text{or } \tan\varphi=\frac{\cos C\cos A+\frac{1}{2}\cos B}{\sin C\cos A-\frac{1}{2}\sin B-\cos\varphi}$$

$$=\frac{m}{n-\cos\varphi}\ldots.[1],$$

where $m=\cos C\cos A+\frac{1}{2}\cos B$, $n=\sin C\cos A-\frac{1}{2}\sin B$.

$$\text{From [1], } n-\frac{m}{\tan\varphi}=\cos\varphi, \text{ whence } n^2-\frac{2mn}{\tan\varphi}+\frac{m^2}{\tan^2\varphi}=\frac{1}{1+\tan^2\varphi}$$

which reduces to

$$\tan^4\varphi-2\frac{m}{n}\tan^3\varphi+\frac{m^2+n^2-1}{n^2}\tan^2\varphi-2\frac{m}{n}\tan\varphi+\frac{m^2}{n^2}=0\ldots.[2].$$

This equation gives one (or more?) lines filling the conditions. By permuting the letters in [1] two more are similarly found. Points X'' and X''' of the figure determine two more lines fulfilling the conditions.

Also demonstrated by G. B. M. ZERR.

173. Proposed by P. C. CULLEN, Principal of Schools, Indianola, Neb.

To construct a circle tangent to a given line at a given point such that tangents drawn to this circle and passing through two fixed points shall be parallel.

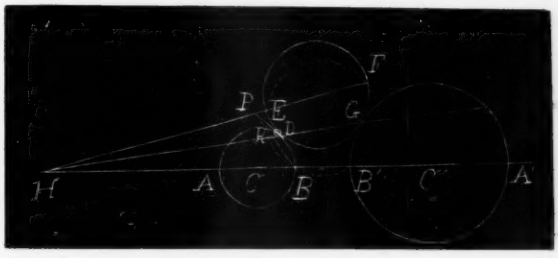
No solution of this problem has been received.

174. Proposed by J.M.HOWIE, Professor of Mathematics, The Nebraska State Normal School, Peru, Neb.

Describe a circle which shall pass through a given point and be tangent to two given circles.

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa., and M. E. GRABER, Heidelberg University, Tiffin, Ohio.

Let C, C' be the centers of the given circles; E the given point. Describe a circle through EBB' and let H be the homothetic center of C, C'. Draw FEH and $B'BH$. Let the circle through $B'BE$ cut C again in R. Draw BRP and from P draw PD tangent to C, then the circle circumscribing the triangle FED is the circle required, for $HD.HG=HB'.HB$. Since there can be two tangents drawn from P to C, there can be drawn two circles tangent to C, C' so that the line joining their points of contact shall pass through H. Similarly, two circles can be drawn tangent to C, C' passing through E so that the line joining their points of contact shall pass through the anti-homothetic center H' lying between C and C'.



Therefore, there are four circles satisfying the condition.

175. Proposed by W. P. WEBBER, Mississippi Normal College, Houston, Miss.

A field is enclosed by a fence in circular form and a straight gate 20 feet wide. The fence is 100 feet in length. How much land is in the field? [Solution by most elementary method possible.]

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; J. SCHEFFER, A. M., Hagerstown, Md.; and the PROPOSER.

Chord $AB=20$, arc $ADB=100$.

Let $\angle ACB=\theta$, $AC=r$.

Then $[2\pi-\theta]r=100$, $r\sin\frac{1}{2}\theta=10$.

$\therefore\ 2\pi=\theta+10\sin\frac{1}{2}\theta$.

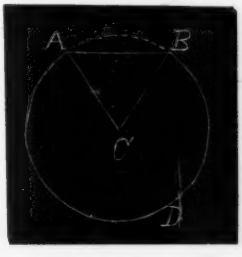
$\therefore\ \theta=62^\circ\ 32'$ nearly, $r=19.267$ feet.

Area sector $ADB=\dfrac{[360^\circ-62^\circ\ 32']\pi r^2}{360^\circ}$

$=963.64$ square feet.

Area triangle $ACB=164.69$ square feet.

Total area$=1128.33$ square feet.



CALCULUS.

133. Proposed by NELSON L. RORAY, South Jersey Institute, Bridgeton, N. J.

Integrate $\int \frac{\sqrt{[1+y]}}{1+y^2}dy$.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let $1+y=z^2$.

$$\therefore \int \frac{\sqrt{[1+y]}}{1+y^2}dy = \int \frac{2z^2dz}{1+[z^2-1]^2} = [1+\sqrt{-1}]\int \frac{dz}{1-\sqrt{-1}+z^2\sqrt{-1}}$$

$$+[1-\sqrt{-1}]\int \frac{dz}{1+\sqrt{-1}-z^2\sqrt{-1}} = -[1-\sqrt{-1}]\int \frac{dz}{1+\sqrt{-1}-z^2}$$

$$-[1+\sqrt{-1}]\int \frac{dz}{1-\sqrt{-1}-z^2} = u.$$

Let $1+\sqrt{-1}=a^2$, $1-\sqrt{-1}=b^2$.

$$\therefore u = -\frac{1}{a^3}\int\left[\frac{1}{a+z}+\frac{1}{a-z}\right]dz - \frac{1}{b^3}\int\left[\frac{1}{b+z}+\frac{1}{b-z}\right]dz$$

$$= \frac{1}{a^3}\log\left[\frac{a-z}{a+z}\right]+\frac{1}{b^3}\log\left[\frac{b-z}{b+z}\right]+C.$$

$$\therefore u = \left[\frac{1}{1+\sqrt{-1}}\right]^{\frac{3}{2}}\log\left[\frac{\sqrt{[1+\sqrt{-1}]}-\sqrt{[1+y]}}{\sqrt{[1+\sqrt{-1}]}+\sqrt{[1+y]}}\right]$$

$$+\left[\frac{1}{1-\sqrt{-1}}\right]^{\frac{3}{2}}\log\left[\frac{\sqrt{[1-\sqrt{-1}]}-\sqrt{[1+y]}}{\sqrt{[1-\sqrt{-1}]}+\sqrt{[1+y]}}\right]+C.$$

Also solved by *F. P. MATZ*. An incorrect solution was received from *H. C. WHITAKER*.

134. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

To find the curve for which the sum of that part of the tangent, lying between the point of contact and the axis of abscissas, and the corresponding ordinate is constant$=c$, and which passes through the point (a, b).

Solution by F. P. MATZ, Sc. D., Ph. D., Defiance College, Defiance, O.; G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; COOPER D. SCHMIDT, A. M., University of Tennessee, Knoxville, Tenn.; and the PROPOSER.

According to the conditions of the problem, *Ordinate* + *Tangent* = *Constant*.

That is, $y+y\sqrt{\left(\frac{dx}{dy}\right)^2+1}=C$....[1].

Therefore, $dx=\frac{\sqrt{[C^2-2Cy]}}{y}dy$....[2].

or, $x=2\sqrt{[C^2-2Cy]}+C\log\left[\frac{\sqrt{[C^2-2Cy]}-C}{\sqrt{[C^2-2Cy]}+C}\right]+C'$....[3].

For $x=a$ and $y=b$, as per the problem, [3] gives the required equation

$$x-a=2\{\sqrt{[C^2-2Cy]}-\sqrt{[C^2-2bC]}\}$$

$$+C\log\left[\frac{\sqrt{[C^2-2Cy]}-C}{\sqrt{[C^2-2Cy]}+C}\Big/\frac{\sqrt{[C^2-2bC]}-C}{\sqrt{[C^2-2bC]}+C}\right]....[4].$$

Also solved by *L. C. WALKER* and *H. C. WHITAKER*. Professor Matz gave a second solution using polar co-ordinates.

135. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

To find the equation of the evolute of the common catenary

$$y=(\tfrac{1}{2}c)(e^{x/c}+e^{-x/c}).$$

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa., and M. E. GRABER, Heidelberg University, Tiffin, O.

$$\frac{dy}{dx}=\tfrac{1}{2}(e^{x/c}-e^{-x/c}),\quad \frac{d^2y}{dx^2}=\frac{1}{2c}(e^{x/c}+e^{-x/c}).$$

$\therefore x-m+(y-n)dy/dx=0$, $1+(dy/dx)^2+(y-n)d^2y/dx^2=0$ become

$$x-m+\tfrac{1}{4}c(e^{2x/c}-e^{-2x/c})-\tfrac{1}{2}n(e^{x/c}-e^{-x/c})=0....(1).$$

$$1+\tfrac{1}{4}(e^{x/c}-e^{-x/c})^2+\tfrac{1}{4}(e^{x/c}+e^{-x/c})^2-\frac{n}{2c}(e^{x/c}+e^{-x/c})=0....(2).$$

From (2), $e^{4x/c}-\frac{n}{c}e^{3x/c}+2e^{2x/c}-\frac{n}{c}e^{x/c}+1=0$.

Let $e^{x/c}=z$.

$\therefore z^4-\frac{n}{c}z^3+2z^2-\frac{n}{c}z+1=0$, $(z^2+1)(z^2-\frac{n}{c}z+1)=0$.

$\therefore z=\frac{n\pm\sqrt{(n^2-4c^2)}}{2c}=e^{x/c}$ and $\frac{n\mp\sqrt{(n^2-4c^2)}}{2c}=e^{-x/c}$.

$\therefore x=c\log\left(\frac{n\pm\sqrt{(n^2-4c^2)}}{2c}\right)$.

These values in (1) give us

$$c\log\left(\frac{n\pm\sqrt{(n^2-4c^2)}}{2c}\right)-m\mp\frac{n}{4c}\sqrt{(n^2-4c^2)}=0,$$

for the equation to the evolute.

136. Proposed by G. B. M. ZERR. A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Evaluate the definite integral

$$\int_0^1\int_0^1 \frac{v^{l-1}u^{m-1}(1-v^n)^{p-1}(1-u^s)^{r-1}dvdu}{[bv^n+c(1-v^n)]^{p+l/n}(u^s+a)^{r+m/s}}.$$

Solution by the PROPOSER.

Let $u^n=z$, $u^s=w$, A=value of integral.

$$\therefore A=\int_0^1\int_0^1 \frac{z^{l/n-1}w^{m/s-1}(1-z)^{p-1}(1-w)^{r-1}dzdw}{[bz+c(1-z)]^{p+l/n}(w+a)^{r+m/s}}.$$

Let $z=\dfrac{cy}{b(1-y)+cy}$, $\quad\dfrac{w}{w+a}=\dfrac{x}{1+a}$.

$$\therefore A=\frac{1}{a^r(1+a)^{m/s}b^{l/n}c^p}\int_0^1\int_0^1 y^{l/n-1}x^{m/s-1}(1-y)^{p-1}(1-x)^{r-1}dydx$$

$$=\frac{1}{a^r(1+a)^{m/s}b^{l/n}c^p}\cdot\frac{\Gamma(l/n)\Gamma(p)\Gamma(m/s)\Gamma(r)}{\Gamma(l/n+p)\Gamma(m/s+r)}.$$

MECHANICS.

133. Proposed by J. C. CORBIN, Superintendent of Schools, Pine Bluff, Ark.

A stick of square-edged timber is 20 feet long, 10 inches square at large end, and 6 inches square at small end. How far from either end must a hand spike be placed so that two men with the hand spike and one man at the end shall each have an equal weight to carry?

Solution by G. B. M. ZERR. A. M., Ph. D., The Temple College, Philadelphia, Pa., and P. H. PHILBRICK. C. E., Lake Charles, La.

Let H=height of pyramid, base 10 inches square; h=height of pyramid, base 6 inches square; z, z_1, z_2 the distances of the centers of mass of the frustum, large pyramid and small pyramid, respectively, from the larger base; m, n the masses of the two pyramids.

$\therefore mz_1=nz_2+(m-n)z$, $z_1=\frac{1}{4}H$, $z_2=\frac{1}{4}h+H-h=H-\frac{3}{4}h$.

Also the masses are to each other as the cubes of the heights.

$\therefore \frac{1}{4}H^4=h^3(H-\frac{3}{4}h)+z(H^3-h^3)$.

$$\therefore z = \frac{H-h}{4}\left(\frac{H^2+2Hh+3h^2}{H^2+Hh+h^2}\right).$$

Let A, a be the lower and upper bases.

$$\therefore z = \tfrac{1}{4}(H-h)\left(\frac{A+2\sqrt{(Aa)}+3a}{A+\sqrt{(Aa)}+a}\right).$$

$H-h=20$ feet, $A=\frac{100}{144}=\frac{25}{36}$ square feet, $a=\frac{36}{144}=\frac{1}{4}$ square foot.
$\therefore z=8\frac{18}{49}$ feet. $20-8\frac{18}{49}=11\frac{31}{49}$.

Taking moments about the center of mass, $2x=11\frac{31}{49}$, $x=5\frac{40}{49}$, $8\frac{18}{49}-5\frac{40}{49}=2\frac{27}{49}$ feet from the larger end.

$2y=8\frac{18}{49}$, $y=4\frac{9}{49}$, $11\frac{31}{49}-4\frac{9}{49}=7\frac{22}{49}$ feet from the smaller end.

134. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

If $pv=Rt-b/tv$ be the equation for CO_2 gas, find the total, external and internal work done in compressing the gas from 102 to 136 atmospheres at a constant temperature 16°C, and constant volume, $R=63.23$, $b=481600$ for CO_2.

Solution by the PROPOSER.

136 atmospheres $=2000$ lbs. $=p_2$, 102 atmospheres $=1500$ lbs. $=p_1$.

$$pv=Rt-b/tv, \text{ or } v=\frac{Rt}{2p}+\frac{1}{2p}\sqrt{\frac{R^2t^3-4bp}{t}}, \quad \frac{dv}{dt}=\frac{R}{2p}+\frac{R^2t^3+2bp}{2pt\sqrt{(R^2t^4-4bpt}}$$

$$=\frac{R}{2p}+\frac{b}{t\sqrt{(R^2t^4-4bpt)}}+\frac{R^2t^2}{2p\sqrt{(R^2t^4-4bpt)}}.$$

$$\text{External work}=t\int_{p_1}^{p_2}\left(\frac{dv}{dt}\right)dp$$

$$=t\int_{p_1}^{p_2}\left(\frac{R}{2p}+\frac{b}{t\sqrt{(R^2t^4-4bpt)}}+\frac{R^2t^2}{2p\sqrt{(R^2t^4-4bpt)}}\right)dp$$

$$=Rt\log_e\left(\frac{Rt^2-\sqrt{(R^2t^4-4bp_2t)}}{Rt^2-\sqrt{(R^2t^4-4bp_1t)}}\right)+\frac{1}{2t}[\sqrt{(R^2t^4-4bp_1t)}-\sqrt{(R^2t^4-4bp_2t)}].$$

Now $t=273+17=290$, $R=63.23$, $b=481600$.
$\therefore$ External work $=18336.7\log_e(1.3367)+46.087=5367.47$ ft. lbs.

$$\text{Total work}=\int_{p_1}^{p_2}vdp=\int_{p_1}^{p_2}\left(\frac{Rt}{2p}+\frac{1}{2p}\sqrt{\frac{R^2t^3-4bp}{t}}\right)dp$$

$$=Rt\log_e\left(\frac{Rt^2-\sqrt{(R^2t^4-4bp_2t)}}{Rt^2-\sqrt{(R^2t^4-4bp_1t)}}\right)+\frac{1}{t}[\sqrt{(R^2t^4-4bp_2t)}-\sqrt{(R^2t^4-4bp_1t)}].$$

$\therefore$ Total work=5275.3 ft. lbs.

$$\text{Internal work}=\int_{p_1}^{p_2}(v-dv/dt)dp=\frac{3}{2t}[\sqrt{(R^2t^4-4bp_2t)}-\sqrt{(R^2t^4-4bp_1t)}]$$

$$=-138.261 \text{ ft. lbs.}$$

Let γ be the ratio of specific heat at constant pressure to specific heat at constant volume; S=dynamic specific heat at constant volume; u=velocity of sound in CO_2, g=gravity, δ=density of mercury, d=density of CO_2, h=height of barometer.

Then $u=\left(\frac{g\delta h[1+(t/273)]\gamma}{d}\right)^{\frac{1}{2}}$. Let $t=0$, then $g=32.2$ feet. $\delta=13.59$, $h=29.92$ inches$=2.4935$ feet, $d=.00198$, u by experiment$=856$ feet.

$\therefore \gamma=du^2/g\delta h=1.3296$.

At 0°C., $S=R/(\gamma-1)=63.23/.3296=191.84$.

For any temperature, t, $S_t=B+t\int_{\infty}^{v}(d^2p/dt^2)dv$.

$dp/dt=R/v+b/t^2v^2$, $d^2p/dt^2=-2b/t^3v^2$.

For $t=273$ absolute temperature and $v=1$ cubic foot, $S_t=191.84=B+2b/t^2v=B+963200/(273)^2$.

$\therefore B=178.943$.

For $t=290°$ absolute or 17°C., $S_t=178.943+963200/(290)^2=190.396$.

$5273.3\div190.396=27.7$°C., the amount the temperature would rise if total work were converted to heat.

135. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

What force acting at an inclination ω with a horizontal line on the center of a wheel of given weight will roll the wheel over an immovable cylindric log whose diameter is $(1/m)$th that of the wheel?

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

If the meaning is to find the force F which will *start* the rolling the following is the solution:

If $\phi=\cos^{-1}\left(\frac{2\sqrt{m}}{m+1}\right)$=the angle the radius of the wheel through the point of contact with the log, and $\phi+\omega=a$, the angle included by this radius and the direction of the required force, and W=the weight of the wheel, we have, taking moments about the point of contact,

$$F.r\sin a=W.\frac{2r\sqrt{m}}{m+1}, \text{ giving } F.$$

DIOPHANTINE ANALYSIS.

90. Proposed by H. S. VANDIVER, Bala, Pa.

Prove that it is always possible to find an infinite number of positive integral values of x, y, and z, such that the relation $z^2=x^2+bxy+cy^2$ is satisfied, b and c being any integers whatever.

II. Solution by J. H. DRUMMOND, LL. D., Portland, Me.

Take $y=mx$, and $z=nx$ and reduce, and we have $n^2=1+bm+cm^2=$(say) $(pm-1)^2=p^2m^2-2pm+1$, and

$$m=\frac{2p+b}{p^2-c} \text{ and } n=\frac{p^2+bp+c}{p^2-c}.$$

Take $x=p^2-c$, and we have $y=2p+b$ and $z=p^2+bp+c$, in which b and c may be any number and p any number that will make p^2 greater than c.

91. Proposed by LON C. WALKER, A. M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

Find the least three positive integral numbers whose sum, sum of their squares, and sum of their cubes shall each be rational squares.

Solution by the PROPOSER.

Let the required numbers by ax, bx, and cx, respectively. Then, by the conditions of the problem, we have

$$(a+b+c)x=\square\ldots.(1),\quad (a^2+b^2+c^2)x^2=\square\ldots,(2), \text{ or}$$
$$a^2+b^2+c^2=\square\ldots.(3); \text{ and } (a^3+b^3+c^3)x^3=\square\ldots.(4), \text{ or}$$
$$(a^3+b^3+c^3)x=\square\ldots.(5).$$

Assume $(a+b+c)x=x^2$; then $a+b+c=x\ldots.(6)$.

Now assume $(a^3+b^3+c^3)x=a^2x^2$; then $x=\frac{a^3+b^3+c^3}{a^2}\ldots.(7)$.

Equating the values of x in (6) and (7), and clearing of fractions,

$$(a+b+c)a^2=a^3+b^3+c^3\ldots.(8).$$

By assuming $a=a-b-c$, substituting in (8), and reducing, we get

$$a^2+(b+c)a=3bc\ldots.(9).$$

From (9), $a=-\frac{1}{2}(b+c)\pm\frac{1}{2}\sqrt{(b^2+14bc+c^2)}\ldots.(10)$.

Now let $b^2+14bc+c^2=\square=(b-\frac{\beta}{\gamma}c)^2$, from which $\frac{b}{c}=\frac{\beta^2-\gamma^2}{2\beta\gamma+14\gamma^2}$.

Hence put $b=\beta^2-\gamma^2$ and $c=2\beta\gamma+14\gamma^2$; then $a=6\beta\gamma-6\gamma^2$.

By substituting these values in (3), and reducing, we get

$$\beta^4+38\beta^2\gamma^2-16\beta\gamma^3+233\gamma^4=\square\ldots.(11).$$

By putting the left member of (11)$=(\beta^2+19\gamma^2)^2$, and solving, we find $\beta=-8\gamma$. To make β positive, let $\beta=\delta-8\gamma$; then substituting in (11), reducing, and putting the left member$=(\delta^2-16\gamma\delta-83\gamma^2)^2$, we find $\delta=1332\gamma/83$, which is an integer when $\gamma=83$. Then $\delta=1332$, $\beta=668$, $a=291330$, $b=439335$, $c=207334$, and $x=937999$. Hence the required numbers are

$$ax=273267248670.$$
$$bx=412095790665.$$
$$cx=194479084666.$$

PROBLEMS FOR SOLUTION.

ARITHMETIC.

160. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

A farm is rented for $\$R,=\$300$, in cash and a certain number of bushels of wheat. When wheat is $\$n,=\$4\text{-}5$, per bushel the rent is $p\%,=12\frac{1}{2}\%$, lower than when wheat is $\$m,=\$1\ 1\text{-}5$, per bushel. Find the number of bushels of wheat.

161. Proposed by F. M. SHIELDS, Coopwood, Miss.

If 1 man, 1 boy and 1 girl catch 1 trout, 1 perch and 1 minnow in 5 minutes, and 1 man, 2 boys and 3 girls catch 1 trout, 2 perch and 3 minnows in 6 minutes, how many minutes will be required for 2 men, 3 boys and 4 girls to catch 5 trout, 11 perch and 17 minnows?

ALGEBRA.

162. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy, Defiance College, Defiance, Ohio.

By Sylvester's dialytic method form the eliminant between $mx^3+py^2=0$ ....(1), and $px^2+my^3=0$....(2). Also between $mx^4+py=0$....(1), and $px^3+my^3=0$....(2).

163. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pa.

Solve $x^4-x=14$, by quadratics.

GEOMETRY.

189. Proposed by J. C. CORBIN, Pine Bluff, Ark.

The perpendicular from the right angle on the hypotenuse of a right-angled-triangle is a harmonic mean between the segments of the hypotenuse made by the point of contact of the inscribed circle. [From Casey's Sequel to Euclid.]

190. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

Find the locus of the centers of sections of an ellipsoid by planes which are at a constant distance from the center.

CALCULUS.

154. Proposed by B. R. DOWNER, Hopkinsville, Ky.

At the equinox, when the sun is on the celestial equator, a man starts driving on a perfectly level plain at six o'clock in the morning, and continues, going always from the sun, at the uniform rate of six miles per hour, until six o'clock in the evening. Required the path he will travel and the distance in a straight line from starting point to stopping point.

155. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Solve the differential equations:

(A). $\frac{d^4y}{dx^4}+2\frac{d^2y}{dx^2}=\sin 2x+\sin x-x.$ (B). $\frac{d^2y}{dx^2}+2\frac{dy}{dx}=\sin 2x+\sin x-x.$

MECHANICS.

144. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Pressure is applied perpendicularly to the plane surface yz, bounding an otherwise infinite isotropic solid. Find the resultant displacements, if the pressure varies as $\sin\left(\frac{2\pi y}{a}\right)+\sin h\left(\frac{2\pi y}{a}\right)$.

145. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

$ABCD$, $GCEF$ are equal parallelograms, DCG and BCE being straight lines. If the figure be considered as formed of smooth light jointed bars and if BD be a light rod, and the whole be suspended from A, find the stress in BD if a weight be hung from F. Also find the stress if a light rod GE replace BD.

DIOPHANTINE ANALYSIS.

104. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

(1). The cube root of three cube numbers equals the square root of two square numbers. Determine the numbers.

(2). The sum of the square roots of three square numbers equals the sum of the cube roots of three cube numbers. Determine the numbers.

105. Proposed by H. S. VANDIVER, Bala, Pa.

Every odd factor of a^n+b^n is of the form $1(\mathrm{mod}2n)$.

AVERAGE AND PROBABILITY.

129. Proposed by J. K. ELLWOOD, Principal of Colfax School, Pittsburg, Pa.

A and B play with two dice, A throwing. If he throws 7 or 11, he wins; if he throws 3, or two aces, or two sixes, B wins. But if he throws 4, 5, 6, 8, 9, or 10, he continues throwing to duplicate this throw, in which event he wins; if in throwing, however, he throws 7, B wins. What is the expectancy of each? [This is the regulation "crap" game, B being banker.]

130. Proposed by L. C. WALKER, A. M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

Four points are taken at random on the surface of a given sphere; show that the average volume of the tetraedron formed by the planes passing through the points taken three and three, is 1-35 of the volume of the given sphere.

NOTES.

Dr. Halsted's article on Non-Euclidean Geometry which was to appear in the March issue of Everybody's Magazine has been unavoidably delayed, so says the editor of that Magazine. But the article is now in type and will soon appear in print.

The mathematicians of the Pacific coast held a meeting in San Francisco on May 3, and organized the second section of the American Mathematical Society, to be known as the Pacific Section. Professor Irving Stringham of California University was elected chairman, and Professor G. A. Miller of Stanford University secretary. The section will hold two meetings per year—in May and December—in or near San Francisco.

The article in the April number of THE MONTHLY on "The Betweenness Assumptions" has called forth some noteworthy comments. Dr. E. H. Moore, of the University of Chicago, writes Dr. Halsted: "I have received from you the April number of THE AMERICAN MATHEMATICAL MONTHLY, containing the proof by Mr. R. L. Moore of the redundancy of Hilbert's Axiom II 4. *The proof is certainly delightfully simple.*" Dr. Moore is so impressed therewith that he has written also to Mr. R. L. Moore: "I read with much interest, the other day, your proof of the redundancy of Hilbert's Axiom II 4, in his system I, II, as exhibited by Professor Halsted in the current number of THE AMERICAN MATHEMATICAL MONTHLY. Today I received from Professor Halsted a copy of that number. This is in response to a letter I sent him a week or so ago stating that I should be pleased to receive for publication in the Transactions the delightfully simple proof of the redundancy of which he wrote to me. I certainly agree with him in this estimate of your proof. * * * I remain with considerable interest in the progress of your mathematical career. Yours very truly, E. H. Moore."

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THE ORDER OF A CERTAIN SENARY LINEAR GROUP.

By PROF. L. E. DICKSON, Ph. D.

In the March number of the MONTHLY, the writer determined the factors of the determinant D of a certain square matrix of order six:

$$(1) \qquad \begin{vmatrix} \mathrm{I} & \alpha & \beta & \gamma & \delta & \varepsilon \\ \beta & \mathrm{I} & \alpha & \delta & \varepsilon & \gamma \\ \alpha & \beta & \mathrm{I} & \varepsilon & \gamma & \delta \\ \gamma & \delta & \varepsilon & \mathrm{I} & \alpha & \beta \\ \delta & \varepsilon & \gamma & \beta & \mathrm{I} & \alpha \\ \varepsilon & \gamma & \delta & \alpha & \beta & \mathrm{I} \end{vmatrix}.$$

It is readily shown that the product of two such matrices is a third matrix of the same form. Hence, if we assign to I, α, β, γ, δ, ϵ all sets of values in a given field, such that D does not equal 0, we obtain a set of matrices having the group property. The group may be represented concretely as a linear homogeneous group in six variables. It is proposed to determine the order of this group in the Galois Field of order p^n, designated $GF[p^n]$. We have only to find the number of sets of elements I, ..., ε such that D does not equal 0. It was shown in the MONTHLY that

$$(2) \qquad D=(\mathrm{I}+\alpha+\beta+\gamma+\delta+\epsilon)(\mathrm{I}+\alpha+\beta-\gamma-\delta-\varepsilon)\triangle^2,$$

$$(3) \qquad \triangle \equiv \mathrm{I}^2+\alpha^2+\beta^2-\mathrm{I}\alpha-\mathrm{I}\beta-\alpha\beta-\gamma^2-\delta^2-\varepsilon^2+\gamma\delta+\gamma\varepsilon+\delta\varepsilon.$$

If $p=3$, Δ is the difference of two squares, so that

$$D\equiv(\mathrm{I}+\alpha+\beta+\gamma+\delta+\varepsilon)^3(\mathrm{I}+\alpha+\beta-\gamma-\delta-\epsilon)^3 \qquad (\mathrm{mod}3).$$

For 3^{5n} sets of values of I, ..., ε in the $GF[3^n]$, the first factor vanishes. Similarly for the second factor. Both vanish simultaneously for 3^{4n} sets of values. Hence the number of sets for which D does not equal 0, *i. e.*, the order of G, is $3^{6n}-(2.3^{5n}-3^{4n})=3^{4n}(3^n-1)^2$.

For p not equal to 3, we set $A=\mathrm{I}-\alpha$, $B=\mathrm{I}-\beta$, $C=\gamma-\delta$, $D=\gamma-\varepsilon$.

Then $\Delta=(A^2-AB+B^2)-(C^2-CD+D^2)$, and the two linear factors of D become

$$f\equiv 3(\mathrm{I}+\gamma)-(A+B)-(C+D),\ f_1\equiv 3(\mathrm{I}-\gamma)-(A+B)+(C+D).$$

The sets I, γ, A, B, C, D for which $D=0$, fall into three classes:

f not equal to 0, f_1 not equal to 0, $\Delta=0$;
f not equal to 0, $f_1=0$; $f=0$,

the second class not occurring if $p=2$, since then $f=f_1$. The third class includes p^{5n} sets. If p is not equal to 2, the second class includes $p^{2n}(p^{3n}-p^{2n})$ sets, since it includes all sets for which

$$3\mathrm{I}-A-B=3\gamma-C-D \text{ not equal to } 0.$$

To determine the number of sets in the first class, let ω be a root of $\omega^2+\omega+1=0$. Since p is not equal to 3, ω^2 is not equal to ω. Then

$$\Delta=(A+\omega B)(A+\omega^2 B)-(C+\omega D)(C+\omega^2 D).$$

If ω belongs to the $GF[p^n]$, $A+\omega B$ and $A+\omega^2 B$ are independent elements of the $GF[p^n]$. Hence there are as many sets A, B, C, D making $\Delta=0$ as there are sets in the $GF[p^n]$ making $xy-zw=0$, viz.,* $p^n(p^{2n}+p^n-1)$.

If ω does not belong to the $GF[p^n]$, we set

$$A+\omega B=X,\ C+\omega D=Y. \tag{4}$$

Since $\omega^{p^n}=\omega$, we have $\Delta=X^{p^n+1}-Y^{p^n+1}$.

We are to determine the number of elements X, Y of the $GF[p^{2n}]$ for which $\Delta=0$. Then $X=RY$ where R is one of the p^n+1 elements for which $R^{p^n+1}=1$. According as Y does not equal 0 or $Y=0$, we obtain $(p^n+1)(p^{2n}-1)$ sets or 1 set X, Y. By (4) each set determines uniquely a set A, B, C, D in the $GF[p^n]$. Hence there are

*This result may also be obtained by subtracting from $(p^n)^4$ the number of sets for which $xy-zw$ does not equal 0, which is the order $(p^{2n}-1)(p^{2n}-p^n)$ of the general binary linear group in the $GF[p^n]$.